# Topic 1

## SEQUENCES AND SERIES

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# 1.1 SEQUENCES AND SUMMATION NOTATION

## 1.1.1 Definition of a sequence

- A **sequence** is a function, *f* whose domain is the set of natural numbers. The terms of the sequence are the function values: $f(1)$, $f(2)$, $f(3)$,…,$f(n)$
- However, the function notation $f(n)$ is commonly written as $a_n$. So, the terms of the sequence are written as such $a_1$, $a_2$, $a_3$,…,$a_n$

# 1.1 SEQUENCES AND SUMMATION NOTATION

## 1.1.2 Recursion formula

- A recursion formula defines the nth term of a sequence as a function that involves one or more of the terms preceding it

Find the first 4 terms of the sequence in which $a_1 = 3$ and $a_n = 2a_{n-1} + 5, n \geq 2$

# 1.1 SEQUENCES AND SUMMATION NOTATION

## 1.1.3 Factorial notation

If *n* is a positive integer, the notation *n!* is the product of all positive integers from *n* down through 1

$$n! = n\,(n-1)(n-2)\ldots\ldots(3)(2)(1)$$

0! by definition is 1

$$0! = 1$$

1) Write the first 4 terms of the sequence whose *n*th terms is $a_n = \frac{20}{(n+1)!}$

2) Evaluate each factorial expression:

a) $\frac{14!}{2!12!}$

b) $\frac{n!}{(n-1)!}$

## 1.1.4 Summation notation

The sum of the first $n$ terms of a sequence is represented by the summation notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots\ldots + a_n$$

where $i$ is the index of summation, $n$ is the upper limit of summation, and 1 is the lower limit of the summation

1) Expand and evaluate the sum:

a) $\sum_{i=1}^{6} 2i^2$ b) $\sum_{k=3}^{5} (2^k - 3)$ c) $\sum_{i=1}^{5} 4$

2) Express each sum using summation notation

a) $1^2 + 2^2 + 3^2 + \ldots\ldots + 9^2$ b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots + \frac{1}{2^{n-1}}$

## 1.1.5 Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences and c is a real number, then

1

- Constant Multiple Rule $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

2

- Sum Rule $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

3

- Difference Rule $$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

4

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{n} a_k \quad when\, 1 < j < n..$$

# 1.1 SEQUENCES AND SUMMATION NOTATION

Example:

1

- Constant Multiple Rule

$$\sum_{i=1}^{6} 2i^2 = 2\sum_{i=1}^{6} i^2$$

2

- Sum Rule

$$\sum_{k=1}^{4} (k + k^3) = \sum_{k=1}^{4} k + \sum_{k=1}^{4} k^3$$

3

- Difference Rule

$$\sum_{k=3}^{5} (2^k - k) = \sum_{k=3}^{5} 2^k - \sum_{k=3}^{5} k$$

4

$$\sum_{k=1}^{8} 3a = \sum_{k=1}^{3} 3a + \sum_{k=4}^{8} 3a$$

# 1.2 ARITHMETIC SEQUENCES

## 1.2.1 Definition of an Arithmetic Sequence

An arithmetic sequence is a sequence of the form

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d, \ldots$$

## 1.2.2 *n*th term of an Arithmetic Sequence

The number $a$ is the first term, and $d$ is the common difference of the sequence. The $n$th term of an arithmetic sequence is given by

$$a_n = a + (n-1)d$$

Find the ninth term of the arithmetic sequence whose first term is 6 and whose common difference is -5

# 1.2 ARITHMETIC SEQUENCES

## 1.2.3 Sum of the 1st $n$th terms an Arithmetic Sequence

Let $\{a_n\}$ be an arithmetic sequence with first term $a$ and common difference $d$. The sum $S_n$ of the first $n$ terms of $\{a_n\}$ is

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

or

$$S_n = \frac{n}{2}[a + a_n]$$

1) Find the sum of the first 15 terms of the arithmetic sequence:

3, 6, 9, 12……..

2) An amphitheater has 50 rows of seats with 30 seats in the first rows, 32 in the second, 34 in the third and so on. Find the total number of seats.

# 1.3 GEOMETRIC SEQUENCES

## 1.3.1 Definition of a Geometric Sequence

A geometric sequence is a sequence in which each term after the first is obtained by multiplying the preceding term by a fixed nonzero constant. The amount by which we multiply each time is called the common ratio of the sequence. A geometric sequence is a sequence of the form

$$a, ar, ar^2, ar^3, \ldots ar^{n-1}, \ldots \; r \neq 0$$

## 1.3.2 *n*th term of a Geometric Sequence

A geometric sequence $\{a_n\}$ whose first term is $a$ and common ratio is $r$, the $n$th term is determined by the formula

$$a_n = ar^{n-1}$$

Example:

Write the first six terms of the geometric sequence with first term 12 and common ratio ½.

# 1.3 GEOMETRIC SEQUENCES

1) Find the seventh term of the geometric sequence whose first term is 5 and whose common ratio is -3

2) Write the general term for the geometric sequence

   3, 6, 12, 24, 48

   Then use the formula for the general term to find the eight term

# 1.3 GEOMETRIC SEQUENCES

## 1.3.3 Sum of the 1st *n*th terms an Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with first term $a$ and common ratio $r$. The sum $S_n$ of the first n terms of $\{a_n\}$ is

$$S_n = a\frac{1-r^n}{1-r}; r \neq 0,1$$

1) Find the sum of the first nine terms of the geometric sequence:

2, -6, 18, -54

2) Find the following sum: $\sum_{i=1}^{8} 2.3^i$

# 1.3 GEOMETRIC SEQUENCES

## 1.3.4 Sum of an infinite geometric series

An infinite sum of the form $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$with first term $a$ and common ratio $r$, and is denoted by

$$S_n = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$ with $-1 < r < 1$

1) Find the sum of the geometric series $3, 2, \frac{4}{3}, \frac{8}{9}, + \ldots$

2) Express $0.\overline{45}$ as a fraction in lowest term.

# 1.4 THE BINOMIAL THEOREM

## 1.4.1 Definition of Binomial Theorem

Binomial Theorom is a formula for the expansion of $(a+b)^n$ for n any positive integer. (Binomial Expansions / Series)

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 2a^1b^3 + b^4$$

# 1.4 THE BINOMIAL THEOREM

## 1.4.2 Definition of a Binomial Coefficient $\binom{n}{r}$

For $n \geq r$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}^{n}C_{r}$$

Evaluate

a) $\binom{6}{3}$ b) $\binom{6}{0}$ c) $\binom{8}{2}$ d) $\binom{3}{3}$

# 1.4 THE BINOMIAL THEOREM

## 1.4.3 A formula or expanding Binomials: The Binomial Theorem

For any positive integer $n$

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + \binom{n}{n}a^0 b^n$$

Expand

a) $(x+1)^4$

b) $(x-2y)^5$

# 1.4 THE BINOMIAL THEOREM

## 1.4.4 General term of a Binomial Expansion

The term containing $a^r$ of the expansion of $(a+b)^n$ is

$$\binom{n}{r} a^{n-r} b^r$$

a) Find the term that contains $x^5$ in the expansion of $(2x+y)^{20}$.

b) Find the coefficient of $x^8$ in the expansion of $\left(x^2+\frac{1}{x}\right)^{10}$.

# References


- Stewart, J., Redlin, L., & Watson, S. (2012) Precalculus: Mathematics for Calculus (6th ed,). Brooks/Cole, Cengage Learning.